

Logarithmic nonlinearity in generally covariant quantum theories: Origin of time and observational consequences

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Abstract

Within the framework of a generic generally covariant quantum theory we introduce the logarithmic correction to the quantum wave equation. We demonstrate the emergence of the evolution time from the group of automorphisms of the von Neumann algebra governed by this non-linear correction. It turns out that such time parametrization is essentially energy-dependent and becomes global only asymptotically - when the energies get very small comparing to the effective quantum gravity scale. We show how the logarithmic non-linearity deforms the vacuum wave dispersion relations and explains certain features of the astrophysical data coming from recent observations of high-energy cosmic rays. In general, the estimates imply that *ceteris paribus* the particles with higher energy propagate slower than those with lower one, therefore, for a high-energy particle the mean free path, lifetime in a high-energy state and, therefore, travel distance from the source can be significantly larger than one would expect from the conventional theory. Apart from this, we discuss also the possibility and conditions of the transluminal phenomena in vacuum such as the “luminal boom” and Cherenkov-type shock waves.

Key words: Quantum gravity - elementary particles - relativity - gamma rays: bursts

PACS numbers: 04.60.Bc, 04.60.Ds, 04.70.Dy, 98.70.Sa

I. INTRODUCTION

In the conventional quantum mechanics the linearity of the wave equation is something which is implicitly presupposed, yet the possibility of the non-linear generalization has not been ruled out by experiment [1]. From the theoretical point of view, there exist arguments that a nonlinearity in general can lead to the violation of locality via the Einstein-Podolsky-Rosen (EPR) apparatus [2]. However, it has been also pointed out that instead of this nonlinearities can lead to communications between branches of the wave function, they can be large in a fundamental theory yet be unobservably small when measured experimentally [3]. Afterwards it was also shown that the locality is not violated for a large class of nonlinear generalizations, including the one which is being discussed here, see Ref. [4], and references therein. In any case, the linearity requirement becomes a rather strict and unnecessary assumption if one expects quantum mechanics to be valid on a wide range of scales: for instance, the modern theory of quantum gravity is believed to be essentially non-linear - because the propagating particle will cause the quantum fluctuations in gravitational medium which will react back.

Some non-linear extensions of QM have been already proposed - for instance, in Ref. [5] authors studied a family of the non-linear wave equations for non-relativistic QM associated with unitary group of certain non-linear gauge transformations of third kind - those which leave the positional probability density invariant. Another general approach to including non-linearity into QM is based on generalizing the quantum phase space to the class of Kähler manifolds which admit certain Hamiltonian flow [6].

On the other hand, those who want to add non-linear terms into the wave equation inevitably arrive at the problem of choice: it seems that there exists a huge variety of the non-linear corrections which can be added without undermining the pillars of the conventional QM - the concepts of the physical state, probability, observables and measurement [7]. Thus, the necessity of non-linear corrections encounters the practicality issue: what are the before unsolved problems of the conventional QM which can be cleared up by introducing non-linearity?

In present paper we do not *ab initio* postulate the linearity of quantum wave equation at all energy scales. Instead we consider the including of one particular nonlinear term in a generally covariant way. We choose the logarithmic one for in the flat-spacetime limit

it would not induce correlations for non-interacting systems [4, 8]. We show that this nonlinearity naturally transforms into the evolution time derivative which becomes global and energy-independent only in the low-energy limit - when the energies are small comparing to the effective quantum gravity scale. Then we discuss the vast observational implications of our model.

II. NON-LINEARITY AND ADDITIVITY

The formal structure of a generally covariant quantum theory is as follows. Let Γ_{ex} be the space of solutions of the generally covariant equations of motion endowed with a degenerate symplectic structure defined by these equations. The degenerate directions of this symplectic structure integrate in orbits and the solutions which belong to the same orbit must be physically identified. The orbits form a symplectic space Γ - a fully covariant object which becomes the physical phase space of the theory. The set $A = C^\infty(\Gamma)$ of the real smooth functions on Γ , called the physical observables, form an Abelian multiplicative algebra. These observables are regarded as classical limits of the non-commuting quantum observables whose ensemble forms the non-Abelian C^* -algebra \mathcal{A} . Since A is a non-Abelian algebra under the Poisson bracket operation one can assume that \mathcal{A} is a deformation of a subalgebra of the classical Poisson algebra. Note that in \mathcal{A} in general there is no defined Hamiltonian evolution or representation of the Poincarè group, therefore, the time evolution is only determined by the dependence of the observables on clock times. Suppose, at the classical level the evolution is governed by the constraint (more precisely, a combination of spacetime diffeomorphisms' constraints) $\mathcal{H} \approx 0$ which vanishes in a weak sense. In the quantum case one must define first a state $\omega: \mathcal{A} \rightarrow \mathbb{C}$, a positive, linear and normed functional on \mathcal{A} . From the Gelfand-Naimark-Segal theorem it follows that there exists exactly one representation $\pi_\omega: \mathcal{A} \rightarrow \text{End}\mathcal{H}_\omega$ of the algebra \mathcal{A} on a Hilbert space \mathcal{H}_ω , and the vector $\xi_\omega \in \mathcal{H}_\omega$ such that: (i) $\text{Lin}(\pi_\omega(\mathcal{A})\xi_\omega) = \mathcal{H}_\omega$, (ii) $\omega(a) = (\pi_\omega(a)\xi_\omega, \xi_\omega)$ for every $a \in \mathcal{A}$ [9]. Then the evolution of physical states is governed by an appropriately chosen operator $\hat{\mathcal{H}}$.

To begin with, by $|\Psi\rangle \in \mathcal{H}_\omega$ we denote the wave functional which describes the state of the dynamical system. Then in the generalized Schrödinger picture the quantum evolution

equation can be written in some representation as (we consider pure states for simplicity):

$$\left[\hat{\mathcal{H}} + F(\Psi) \right] \Psi = 0, \quad (1)$$

where the first term in brackets is essentially the above-mentioned combination of constraints \mathcal{H} quantized as in the conventional formalism. Its explicit form is determined by a concrete physical setup and thus will not be important for us here - we simply assume that it can be consistently defined. This will make our following results largely model-independent from spacetime-formulated theories. The other term, $F(\Psi)$, is not present in the conventional quantization procedure. As to preserve the probabilistic interpretation of Ψ (the physical states are actually not vectors but rays), we assume that F depends not on Ψ alone but rather on its complex square. Notice also that this term does not interfere with \mathcal{H} as it describes the self-interaction of the wave functional and thus is inherent only to the way we define the quantum wave equation. Therefore, we write Eq. (1) as

$$\left[\hat{\mathcal{H}} + F(\rho) \right] \Psi = 0, \quad (2)$$

where $\rho \equiv |\Psi|^2$. What is the explicit form of the operator \hat{F} ?

Suppose the system described by Ψ consists of two separated distinguishable subsystems, described by wave functionals Ψ_1 and Ψ_2 , respectively, which obey the wave equations $\left[\hat{\mathcal{H}}_i + F(\rho_i) \right] \Psi_i = 0$, $i = 1, 2$, where we denoted $\rho_i \equiv |\Psi_i|^2$. We assume that F has the form

$$F(\rho) = -\beta^{-1} \ln(\Omega \rho), \quad (3)$$

where β and Ω are arbitrary positive constants with the dimensionality of the inverse energy and space volume, respectively. This function is a general solution of the following algebraic equation

$$F(\rho_1 \rho_2) = F(\rho_1) + F(\rho_2), \quad (4)$$

if β has the same value for both subsystems, therefore, for two uncorrelated subsystems, when the wave functional of a whole system becomes the product $|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle$, the overall quantum wave equation turns to: $\left[\hat{\mathcal{H}}_{12} + F(\rho_1) + F(\rho_2) \right] (\Psi_1 \Psi_2) = 0$, thus the non-linear part in this special case becomes a plain sum of the subsystems' ones. Therefore, the non-linear term obeying Eq. (4) introduces on its own no additional correlations between the uncorrelated subsystems for which β takes the same value. Later it will be shown that,

for instance, in the weak-gravity limit the relative difference between any two β 's is given by $\Delta\beta/\beta \sim \Delta E/E_{\text{QG}}$, where E_{QG} is certain very large energy scale and $\Delta E \ll E_{\text{QG}}$ is the difference of the subsystems' energies, and therefore, in that limit Eq. (4) holds with a high degree of precision. Thus, the logarithmic non-linearity on its own does not break the energy additivity and separability of non-interacting subsystems in non-relativistic quantum mechanics. Therefore, from the viewpoint of the correspondence principle the logarithmic term is safe to include into the quantum wave equation.

Gathering all together, from Eqs. (2) and (3) in the position representation we obtain the following quantum wave equation:

$$\left[\hat{\mathcal{H}} - \beta^{-1} \ln(\Omega |\Psi(x)|^2) \right] \Psi(x) = 0, \quad (5)$$

which is the quantized version of the constraint $\mathcal{H} \approx 0$ in generally covariant theories. This equation can be formally written as

$$\hat{\mathcal{H}}' |\Psi\rangle \equiv \left(\hat{\mathcal{H}} + \beta^{-1} \hat{S}_\Psi \right) |\Psi\rangle = 0, \quad (6)$$

where \hat{S}_Ψ is the Hermitian operator defined in the following way [10]: let us consider the operator \hat{S}_f labeled by f , f is an arbitrary square-integrable and nowhere-vanishing function, which satisfies the equation $\hat{S}_f |\Psi\rangle = -\ln(\Omega |f|^2) |\Psi\rangle$. It is easy to see that \hat{S}_f is a quantum-mechanical operator in the conventional sense. Then the operator \hat{S}_Ψ is defined as \hat{S}_f evaluated on the surface $f - \langle x | \Psi \rangle = 0$. This implies the use of the projective Hilbert space but the latter is not a problem because even in the conventional quantum formalism the space of physical states is already a projective Hilbert space (rays), due to the normalization constraint. Of course, to preserve interpretation of Ψ as a wave function, it must be normalizable and also the corresponding probability density must obey the conservation law which is the case for the logarithmic nonlinearity [8]. Otherwise, the nonlinear quantum wave equation can be interpreted only as an effective one, one example to be the Gross-Pitaevskii equation (yet the term “effective” should be treated with care here because in quantum gravity, unlike condensed matter, the background medium cannot be eliminated).

To summarize, the nonlinear term (3) has a number of physically meaningful distinctive properties which make it a very probable candidate for a nonlinear correction to the quantum wave equation: (i) it would not break the general covariance of a theory because it does not interfere with the spacetime diffeomorphisms' constraints (yet, of course, it takes them into

account - through the dependence on the wave functional which, in turn, is a solution of the wave equation), (ii) it does not violate locality via EPR-type correlations, as discussed above [4], (iii) if wave function satisfies Eq. (5) then this function times any constant is also a solution provided that $\hat{\mathcal{H}}'$ is shifted by a constant, (iv) symmetry properties of wave functions with respect to permutations of the coordinates of identical particles are not affected by the nonlinear term, (v) in the non-relativistic limit the equations for the probability density and current are not altered, energy is additive and bounded from below, free-particle waves are localized. The non-linear corrections of other kind can also be included into Eq. (6) but they would violate these properties, therefore, in what follows we focus only on the non-linear term (3). Below it will be shown that it has some additional properties which make it very useful in quantum gravity theories.

III. MODULAR GROUP AND EVOLUTION TIME

The physical meaning of the non-linear term in Eq. (6) becomes a bit more clear when we go to the non-relativistic limit. Its averaging in the position basis yields

$$\langle \beta^{-1} \hat{S}_\Psi \rangle \equiv -\beta^{-1} \int |\Psi|^2 \ln(\Omega |\Psi|^2) d^3x \equiv T_\Psi S_\Psi, \quad (7)$$

where the functional S_Ψ is the Shannon-type entropy which can be formally assigned to a single quantum particle, $T_\Psi \equiv (k_B \beta)^{-1}$, k_B is the Boltzmann constant. From the analysis of its discrete counterpart one can infer that S_Ψ reaches a minimum on the delta-like probability distribution (which corresponds to a classically localized particle) and maximizes on the uniform one [10]. This is what the information entropy usually does, therefore, S_Ψ can be interpreted as a measure of the particle's "smearing" over space and corresponding quantum uncertainty. Such quantum-mechanical entropy is a purely information one and should not be confused with the von Neumann entropy - the latter would vanish for pure states and thus neglect an inherent quantum-mechanical uncertainty of the outcome of a measurement [11]. Perhaps, the notion of entropy closest to S_Ψ would be the one proposed by Wehrl for the coherent states [12].

Further, strictly speaking, Eq. (6) contains no evolution time parameter. It is the well-known property of fully covariant theories that \mathcal{H} contains the geometrical time-like coordinate at most but the dynamics cannot be formulated in terms of a single external

time parameter. This creates the conceptual difficulties because the full dynamics of the quantum gravity theory can not be consistently defined in terms of other available notions of time, such as the proper time, because they are state-dependent [13]. They can serve as the evolution times only for dynamical theories on a *fixed* spacetime geometry but not for the dynamical theory of geometry which the quantum gravity is supposed to be.

One way to define the evolution time without invoking assumptions for spacetime geometry is to formulate it based on notions of a statistical and thermodynamical nature. To our knowledge, the initial idea was proposed in Ref. [14] - to define the evolution time as the vector flow of the Gibbs state on the constraint surface. Is there any way to derive the notion of the evolution time from the quantum wave equation itself? We showed before that the nonlinear term from Eq. (6) can be interpreted as a kind of entropy, at least in the non-relativistic limit. Our next step will be to show that this term gives rise also to the evolution time in quantum theory.

Luckily, the mathematical background for justifying this has been already developed, both for the conventional spacetime geometry [13] and for the non-commutative one [15]. Recalling the notations above, let \mathcal{R} be the von Neumann algebra generated by the representation of \mathcal{A} , $\pi_\omega(\mathcal{A})$, on the Hilbert space. Then the Tomita-Takesaki theorem asserts that the mappings of the von Neumann algebra on itself, $\alpha_\tau: \mathcal{R} \rightarrow \mathcal{R}$ ($\tau \in \mathbb{R}$), of the form $\alpha_\tau(b) = \Delta^{-i\tau} b \Delta^{i\tau}$, $b \in \mathcal{R}$, where Δ is a self-adjoint positive operator, form a one-parameter group of automorphisms of \mathcal{R} , called the modular group of the state ω . This can be equivalently written as

$$\dot{b} \equiv \frac{d}{d\tau} \alpha_\tau(b)|_{\tau=0} = i[b, \ln \Delta], \quad (8)$$

thus, the “modular” time α_τ can be already regarded as the time we are needed in but it seems still state-dependent so a bit of final tuning must be made. Some of automorphisms on \mathcal{R} are inner-equivalent, the set of their equivalence classes forms a group of outer automorphisms $\text{Out}(\mathcal{R})$. Then the modular group α_τ of a state ω projects down to a nontrivial group $\tilde{\alpha}_\tau \in \text{Out}(\mathcal{R})$. The state-independent characterization of time is then proven by the cocycle Radon-Nykodym theorem from which it follows that $\tilde{\alpha}_\tau$ does not depend on choice of ω in \mathcal{R} .

Further, from Eq. (6) we obtain

$$0 = [b, \hat{\mathcal{H}}'] = [b, \hat{\mathcal{H}}] + \beta^{-1}[b, \hat{S}_\Psi], \quad (9)$$

for any $b \in \mathcal{R}$. Then, observing that the product $e^{i\tau\mathcal{H}'} b e^{-i\tau\mathcal{H}'}$ is equivalent to $e^{i\tau S_\Psi/\beta} b e^{-i\tau S_\Psi/\beta}$ on the constraints' surface, we can assume that

$$\Delta \propto \exp \hat{S}_\Psi, \quad (10)$$

from which, using Eq. (8), we obtain the equation of motion in the generalized Heisenberg picture:

$$\frac{i}{\beta} \frac{d}{d\tau} b = [\hat{\mathcal{H}}, b], \quad (11)$$

where the derivative is understood in a sense of the commutator from Eq. (8) with the generator given by Eq. (10). This essentially means that the dynamics described by the “stationary” operator $\hat{\mathcal{H}}'$ is equivalent to the evolution governed by $\hat{\mathcal{H}}$ with respect to the evolution time $\beta\tau$. In other words, the logarithmic nonlinearity can be “used up” for creating the time evolution of a generally covariant theory: theory containing such nonlinearity but without the observer-independent time evolution is equivalent to the *linear* theory with the evolution time defined by the modular group. As shown below, only in the low-energy limit this time parametrization becomes global.

IV. DISPERSION RELATIONS AND OBSERVATIONAL TESTS

From Eq. (6) one can see that the dispersion relation for a particle in vacuum is being deformed by the non-linear term. This is not surprising though as the quantum gravity is expected to give rise to such corrections [16–18] because the gravitational medium contains quantum fluctuations which respond differently to the propagation of particles of different energies - the phenomenon somewhere analogous to propagation through electromagnetic plasmas [19]. The full treatment of this problem is impossible without establishing a concrete quantum gravity model. Yet, some features are model-independent and can be clarified already at this stage.

First, despite the functional form of the non-linear term is universal for all dynamical systems the constant β is not a fundamental one hence depends on dynamical characteristics of a system. To find its physical meaning, we go to the flat-spacetime limit and consider a norm-preserving splitting of an arbitrary wave function into N non-overlapping parts of the same form as the initial wave function: $\Psi(x) \rightarrow \sum_{i=1}^N \sqrt{p_i} \Psi(x - x_i)$ where $\sum_{i=1}^N p_i = 1$. Then

from the averaged Eq. (6) one obtains that the change of energy during the process amounts to $\delta E = -\beta^{-1} \sum_{i=1}^N p_i \ln p_i$, and thus β is a measure of the binding energy,

$$\beta \propto 1/\delta E. \quad (12)$$

Second, from Eq. (11) one can immediately see that for any two dynamical systems the relation

$$\beta_1 d\tau_1 = \beta_2 d\tau_2, \quad (13)$$

must hold, thus, if two systems have different β 's then their evolution time scales must differ as well.

Now, suppose that some two particles are products of the reactions happened inside a compact region of space. In the process these particles receive certain amounts of energy, E_1 and E_2 , respectively. From the previous two equations we obtain that the ratio of their evolution time scales is given by

$$\frac{d\tau_2}{d\tau_1} = \frac{\beta_1}{\beta_2} = \frac{E_2 - E_0}{E_1 - E_0} = 1 - \frac{E_2 - E_1}{E_0} + \mathcal{O}(E^2/E_0^2), \quad (14)$$

at least in the leading-order approximation. Here E_0 is the energy of vacuum of a theory, we imply that $|E_0| = E_{\text{QG}}$ where $E_{\text{QG}} \lesssim 10^{19}$ GeV is the effective quantum gravity energy scale. The large value of the latter explains why our (non-relativistic) notion of time is global and energy-independent: in the low-energy regime the characteristic energies of any two particles become small comparing to the quantum gravity energy scale, therefore, the difference between any two β 's also gets vanishingly small, $(\beta_2 - \beta_1)/\beta_i \sim (E_2 - E_1)/E_{\text{QG}}$, and when the value of β is essentially the same for all dynamical systems then this constant can be absorbed into the time parameter. Thus, the reason why the logarithmic nonlinearities are not observed in current quantum-mechanical experiments is not because of their smallness but because they act as the time derivatives with individual scale factors which are essentially indistinguishable in the low-energy regime.

Another observation can be made if one recalls that the correspondence principle implies that for elementary particles their “modular” times must synchronize with their proper times in the weak-gravity limit. Then Eq. (14) immediately reveals the presence of the Lorentz invariance violation (LIV): it is not the conventional line element of spacetime which is invariant but the one multiplied by an energy-dependent function [20]. However, in the low-energy limit this circumstance does not bring any phenomenological difficulties because for a

conformally flat spacetime (such as the Friedmann-Lemaître-Robertson-Walker one) one can still define the representations of the Lorentz group in the vierbein basis, therefore, the main notions of particle physics are preserved [21]. In any case, LIV is an expected phenomenon in quantum gravity - the nontrivial vacuum creates a preferred frame of reference.

Further, our particles fly off, travel across the space and eventually get caught by a remote detector. If they are of same kind, initially were emitted approximately simultaneously with equal velocities and their travel conditions were similar then what kind of differences between them are our detectors supposed to catch? Once again, the exact quantitative results are impossible without employing the concrete model of quantum gravity yet some heuristic analysis can be done in the flat-spacetime approximation (the cosmological corrections are not considered for now as to avoid certain confusions).

Making in Eq. (14) the transition from proper time to the distant observer coordinate time t (with cosmological effects that would be the comoving time), we obtain

$$\frac{v_1}{v_2} \sqrt{\frac{c^2 - v_2^2}{c^2 - v_1^2}} = \frac{E_2 - E_0}{E_1 - E_0}, \quad (15)$$

where $v_i = dx/dt_i$. This equation reveals the following subtlety: if for our future purposes we assume that the particles are essentially relativistic or even ultrarelativistic and also that their velocities are nearly the same, then the value of a square root in the equation above crucially depends on whether the ratio v_1/c tends to unity “stronger” than v_1/v_2 .

Thus, there exist two limit regimes of analysis: *linear* or standard relativistic - when the ratio v_1/v_2 approaches one “stronger” than v_1/c does, and *non-perturbative* or extreme ultrarelativistic - when it is other way around. When analyzing concrete experimental data, one should look at these two ratios to decide which regime s/he is next to. Especially one should be careful when the cosmological effects are taken into account because these ratios may vary as the particles propagate.

A. Linear regime

In this case the square root in Eq. (15) can be well approximated by one, therefore, under the above-mentioned assumptions we obtain:

$$\frac{v_1}{v_2} \approx \frac{E_2 - E_0}{E_1 - E_0}, \quad (16)$$

thus, under equal conditions the particle with lower energy travels faster, so we can write $v \sim v^{(0)}/(1-E/E_0)$ where $v^{(0)} = v_{E/E_0=0}$. The difference in their arrival times is proportional to their energy difference:

$$t_2 - t_1 \approx t_1 \frac{E_2 - E_1}{E_1 - E_0}. \quad (17)$$

For instance, for the photons we obtain $\Delta t \approx \frac{L}{cE_{\text{QG}}} \Delta E$, where L is the distance from a distant observer to the actual place where the reactions happened. This is what is being often observed about cosmic ray photons coming from the very distant Gamma-ray bursts (GRB), the highly energetic explosions of massive stars in galaxies: in the linear approximation the difference in the arrival times of photons is proportional to their energy difference. For instance, during the exceptionally luminous GRB 080916C, distant from us as far as 12.2 billions light years, the first photons with energies above 1 GeV started to arrive only after ten seconds after the trigger, e.g., the 13.2 GeV photon had arrived after 16.54 s (the duration of the whole event itself was few tens seconds) [22]. The cosmological-scale remoteness of this and some other GRB's plays a crucial role here: from the last formula it is clear that L/c should be very large so as to win over the huge number $E_{\text{QG}}/\Delta E$ and produce an experimentally detectable effect.

The observational predictions can also be formulated on language of the (deformed) dispersion relations for particles *in vacuo*. From Eq. (16) we obtain

$$\frac{\Delta v}{\Delta E} \equiv \frac{v_2 - v_1}{E_2 - E_1} = \frac{v_2}{E_0 - E_2}, \quad (18)$$

therefore, if $E \ll |E_0| \sim E_{\text{QG}}$ then $dv/dE \approx \Delta v/\Delta E \sim -\xi v/E_{\text{QG}}$, where we assume $\xi \equiv -\text{sign}(E_0) = \pm 1$. It means that the velocity can be written as an exponent of energy and thus is a linear function of E in the leading order:

$$v/c \sim \exp(-\xi E/E_{\text{QG}}) = 1 - \xi \frac{E}{E_{\text{QG}}} + \mathcal{O}(E^2/E_{\text{QG}}^2), \quad (19)$$

and same result can be obtained if we look for a Taylor series solution of Eq. (16): $v_i/c = \sum_{n=0}^N a_n (E_i/E_0)^n$, where $N \geq 1$ is an approximation order, a_n are energy-independent constants to be determined. The boundary conditions are determined by physics - for instance, for the case of photons they would be: $v_i = c$ when $E_i/E_0 \rightarrow 0$.

With Eq. (19) in hands we can immediately recover the results of Refs. [18, 23, 24]. Indeed, the resulting dispersion relation for the rotationally invariant case, $\partial E/\partial p = v(E)$, integrates to

$$c^2(p - p_0)^2 \sim E_{\text{QG}}^2 (1 - \xi e^{\xi E/E_{\text{QG}}})^2 = E^2 \left[1 + \xi \frac{E}{E_{\text{QG}}} + \mathcal{O}(E^2/E_{\text{QG}}^2) \right], \quad (20)$$

where p_0 is an integration constant. In the limit $p_0 \rightarrow 0$ the relation reduces to the dispersion relation for massless particles in the effective quantum gravity theories based on κ -deformations of Poincaré symmetries in which the time coordinate does not commute with spatial ones [25, 26]. On the other hand, with the appropriate choice of the integration constant the last equation can be approximated by (in high-energy units): $E^2 \simeq p^2 + m^2 - (E/E_{\text{QG}})p^2$, from which one can find out that the kinematics of particle-production processes (such as the photopion production $p + \gamma \rightarrow p + \pi$, etc.) will be affected - the photopion-production threshold energy gets increased by the deformation $E > \frac{(2m_p + m_\pi)m_\pi}{4\epsilon} \left[1 + \frac{(2m_p + m_\pi)^2 m_\pi^2}{64\epsilon^3 E_{\text{QG}}} \left(1 - \frac{m_p^2 + m_\pi^2}{(m_p + m_\pi)^2} \right) \right]$, where E and ϵ are the energies of a proton and cosmic microwave background photon, respectively, m_p and m_π are proton's and pion's masses. Similarly, for the electron-positron pair production process $\gamma + \gamma \rightarrow e^- + e^+$ one obtains the threshold $\mathcal{E} > \frac{m_e^2}{\epsilon} + \frac{m_e^6}{8\epsilon^4 E_{\text{QG}}}$, where \mathcal{E} is the energy of a traveling photon, m_e is the electron mass.

B. Non-perturbative regime

As was mentioned earlier, if the propagation speed of a particle is very close to c then one can not approximate Eq. (15) by Eq. (16), and therefore, the expression (19) can not be valid in general.

There exist at least three ways of how one can derive here the correct expression for velocity as a function of energy. First way is to assume $v_2 = v_1 + \Delta v$, $E_2 = E_1 + \Delta E$ in Eq. (15), expand the latter to a linear order w.r.t. Δv , replace therein Δ 's by their infinitesimal counterparts and integrate in the spirit which led us to Eq. (19). Second way is to look for an approximate solution of Eq. (15) in a series form $v_i/c = \sum_{n=-2N}^{2N} a_n(\epsilon_i)^n$, where $\epsilon_i \equiv \sqrt{E_i/E_0} \ll 1$ ($E_0 \equiv -\xi E_{\text{QG}}$, as before, $\xi = \pm 1$), $N \geq 1$ is an approximation order, a_n are energy-independent constants to be determined. The third method [27] might look not as rigorous as the previous two but it is fast and elegant: one should just write Eq. (15) in the form $\sqrt{\frac{(c/v_2)^2 - 1}{(c/v_1)^2 - 1}} = \frac{1 - E_2/E_0}{1 - E_1/E_0}$, where its solution becomes obvious.

All these methods lead to the same result: the desired $v(E)$ is a solution of the algebraic

equation $\sqrt{(c/v)^2 - 1} = \sqrt{\chi^2 - 1}(1 - E/E_0)$, namely

$$v/c = \left[1 + (\chi^2 - 1) \left(1 - \frac{E}{E_0} \right)^2 \right]^{-1/2}, \quad (21)$$

where $\chi = (c/v)_{E/E_0=0}$ is the emerging parameter which value can not be determined from Eq. (15) alone. By construction this parameter does not depend on energy of a particle but may vary for different kinds of particles. From this expression one can directly computed the effective refractive index of the vacuum. In the Cauchy form it can be written as [28]

$$n^2 \equiv (c/v_\gamma)^2 = 1 + \mu_\gamma [1 + \mathcal{M}(\omega)(\omega/2\pi c)^2], \quad (22)$$

where $\mu_\gamma = \chi_\gamma^2 - 1$ and $\mathcal{M}(\omega) = (2\pi c/\omega_0)^2 (1 + 2\xi\omega_0/\omega)$ are, respectively, the constant of refraction and dispersion coefficient of the physical vacuum, v_γ is the velocity of a photon, ω is the angular frequency of the electromagnetic wave, $\omega_0 = |E_0|/\hbar$ is the natural frequency of the vacuum. All this confirms once again that the physical vacuum is the medium with non-trivial properties which affects photons and other particles propagating through it, and the effects grow along with particles' energies.

The final dispersion relation can be obtained by subsequent integration, as for Eq. (20):

$$p - p_0 = -\frac{E_0}{2c\sqrt{\mu}} \left[\Upsilon(\sqrt{\mu}(1 - E/E_0)) - \operatorname{arcsinh}(\sqrt{\mu}) - \chi\sqrt{\mu} \right], \quad (23)$$

where we denoted $\Upsilon(x) \equiv x\sqrt{1+x^2} + \operatorname{arcsinh} x$, $\mu \equiv \chi^2 - 1$, and p_0 is the integration constant representing the momentum of the background, it is convenient to work in the comoving frame of reference where $p_0 = 0$. It can be convenient also to eliminate χ from Eqs. (21) and (23) to obtain the expression for momentum as a function of energy and velocity:

$$p = \frac{E_0 - E}{2c\Gamma} \left[\Upsilon(\Gamma/(1 - E/E_0)) - \operatorname{arcsinh}(\Gamma) - \Gamma\sqrt{1 + \Gamma^2} \right], \quad (24)$$

where by $\Gamma \equiv \sqrt{(c/v)^2 - 1}$ we denoted the inverse Lorentz factor. In fact, this formula is the replacement of the relativistic dispersion relation $p = Ev/c^2$ for ultrarelativistic particles with high energies $E \not\ll |E_0|$.

The main feature of the non-perturbative solution is that it indicates the existence of the different classes or sectors, depending on the value of χ . However, unlike the classical relativity, sectors of the “subluminal” ($v \leq c$) and “luminal” ($v = c$) particles are not totally disconnected: the propagation speed of the subluminal particles can reach c at *finite* energy. Among other things, this may cause the transluminal phenomena in vacuum discussed below.

First mode, called standard or analytic, can be seen when $\chi \neq 0$. In this case Eq. (21) allows expansion into the Taylor series w.r.t. energy:

$$\frac{v^{(s)}}{c_\chi} = 1 + \frac{\chi^2 - 1}{\chi^2} \frac{E}{E_0} + \frac{(\chi^2 - 1)(\chi^2 - \frac{3}{2})}{\chi^4} \frac{E^2}{E_0^2} + \mathcal{O}(E^3/E_0^3), \quad (25)$$

where $c_\chi \equiv c/\chi$ is the “renormalized” speed of light. This shows that for photons in this mode the inverse of χ can be interpreted as the “luminal” Mach number and thus χ is related to the (effective) refractive index of the physical vacuum. To prevent their motion from becoming superluminal in this mode, χ^2 must be larger than 1 - but not much larger, most probably no more than ten percent, as to retain the formal value of c_χ close to c . This makes the dimensionless series coefficients in Eq. (25) small - in addition to the smallness of the ratio E/E_0 itself.

Another mode, called anomalous, is given by the non-analytic branch of the solution (21) at $\chi = 0$:

$$v^{(t)} = \frac{c/\sqrt{2}}{\sqrt{E/E_0}} \left[1 + \frac{1}{4} \frac{E}{E_0} + \mathcal{O}(E^2/E_0^2) \right], \quad (26)$$

of course, this expression can be valid when $v \gtrsim c$ where energy of a particle can not be vanishing. For ultrarelativistic particles there is no obvious boundary condition to rule such modes out, therefore, this mode is the (particular example of the) “superluminal” one: it is an essentially LIV phenomenon which describes a particle which can propagate in the (nontrivial) vacuum with the velocity larger than c .

Generally speaking, the superluminal modes exist for $\chi^2 < 1$, as one can directly see from Eq. (21). Unlike the tachyons in the classical relativity theory, their energies are real-valued and stay finite when v approaches c . If we extend χ on to the complex plane (but keeping its square real-valued) then the superluminal particles can be further classified depending on whether χ^2 is a strictly positive number or not. If it is then the minimal allowed energy of such particles is zero, otherwise, i.e., when χ is imaginary or zero, the energy must be greater than some threshold value: $E_{\min} \equiv E_0(1 - 1/\sqrt{1 - \chi^2})$. Thus, they are not expandable in series in the vicinity $E = 0$. The mode (26) is a special case of the second subclass and is a kind of the “interface” mode between the subclasses: its E_{\min} is zero (similarly to the first subclass) but $v(E)$ is not analytic in that point (similarly to the second subclass).

The common feature of the superluminal particles is for the “sub-Planckian” energies ($|E/E_0| < 1$) their propagation speed decreases as energy increases - as opposite to the subluminal mode (25) - until it reaches c . This can be explained by when a particle propagates

faster than the speed of light in the physical vacuum its interaction with the latter leads to the “luminal boom” and appearance of a conical front of the shock waves carrying away large amount of energy. In the classical relativity this energy is actually infinite and thus the barrier crossing would be forbidden for known particles. This Cherenkov-type radiation is mostly electromagnetic but it can lead to creation of other known particles - often with very high energies. An interesting question is whether the electromagnetic component of this radiation exhibit the anomalous Doppler effect - similar to the one for the superluminal (non-point) sources in vacuum which has been predicted even at the level of the classical relativity theory [29].

From the observational point of view, so far there exist not so much arguments directly supporting the existence of luminal booms in the (nontrivial) vacuum in some GRBs and AGNs - probably because one can often find more than one way of explaining the majority of astrophysical phenomena, especially if a phenomenon is complex and/or the observational data lack of a necessary accuracy. Nevertheless, some arguments do exist: for instance, the softening of a GRB afterglow is similar to the frequency evolution of a sonic boom: at the surface of the shock cone the frequency is very high but rapidly decreases inside, and there is even some quantitative similarity [30]. Also there may appear the phenomenon of mimicking the double-lobed radio sources, such as DRAGNs, by such shock waves - by analogy with any two sound waves from a supersonic jet initially emitted at different times (and thus from different locations) but reached an observer simultaneously thus creating an illusion of the doubling of the sound source. In this connection, another interesting question is what would be the “luminal” analogues of other trans- and supersonic phenomena, such as the Prandtl-Glauert singularity, N- and U-wave, *etc.*

To summarize, all this reasoning which led us from Eq. (14) to Eqs. (19) and (21) indicates that in our theory the dispersion relation for all scales of energy and momentum may actually vary depending on a physical situation, and therefore, the complete physical picture is still on its way - main reason of which was explained in the paragraph preceding Eq. (15). In this connection, it would be interesting to find the way of observational testing of directly Eq. (14), or Eq. (15) but taking into account cosmological effects, as they are the primary predictions of the theory.

V. CONCLUSION

In general, our estimates imply that due to the effects of the nontrivial physical vacuum the mean free path of a subluminal high-energy particle, its lifetime in a high-energy state and, therefore, travel distance from the source can be significantly larger than one would expect from the conventional theory. In fact, using arguments of such kind one can show that the deformed dispersion relations above are capable of explaining results of few other classes of experiments: observations of cosmic rays above the expected GZK limit, studies of the longitudinal development of the air showers produced by ultra-high-energy hadronic particles, ATIC observations of the high-energy electrons from an unseen source [23, 24, 31–33]. The trans- and superluminal phenomena are briefly discussed as well.

Acknowledgments

I acknowledge the fruitful discussions with A. Avdeenkov, M. Kastner, F. Petruccione, M. Pörrmann, F. Scholtz, I. Snyman, E. Tkalya and S. Vaidya, as well as the useful correspondence from M. Czachor, N. Mavromatos, V. Mitsou and F. Stecker. I am grateful also to C. Chryssomalakos, H. Quevedo and D. Sudarsky from the Universidad Nacional Autónoma de México who introduced me into the field of effective quantum gravity some time ago. My research was partially supported by grant 68523 from the National Research Foundation of South Africa. Last, but not least, I thank the National Institute of Theoretical Physics (NITheP) and Stellenbosch Institute of Advanced Studies (STIAS) and their Directors, F. Scholtz and H. Geyer, for hospitality.

Remarks. After (the initial version of) our paper has been e-printed we found that in the very recent article, “A limit on the variation of the speed of light arising from quantum gravity effects” by the Fermi LAT and GBM Collaborations, published in Nature 462, pp 331–334 (e-printed at arXiv:0908.1832), the velocity dispersion (19) has been ruled out. For our theory it is not a problem though because in that particular case the linear approximation described in Sec. IV A is hardly valid. For the extremely ultrarelativistic particles, one should consider instead the non-perturbatively derived dispersion from Sec. IV B, eventually

leading to Eq. (25). The latter is not ruled out by Fermi’s data - those can only put further bounds for the parameter χ .

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